Entanglement Swapping Chains for General Pure States

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Abstract

We consider entanglement swapping schemes with general (rather than maximally) entangled bipartite states of arbitary dimension shared pairwise between three or more parties in a chain. The intermediate parties perform generalised Bell measurements with the result that the two end parties end up sharing a entangled state which can be converted into maximally entangled states. We obtain an expression for the average amount of maximal entanglement concentrated in such a scheme and show that in a certain reasonably broad class of cases this scheme is provably optimal and that, in these cases, the amount of entanglement concentrated between the two ends is equal to that which could be concentrated from the weakest link in the chain.

1 Introduction

There are many applications of quantum entanglement including quantum cryptography [1, 2, 3], teleportation [4] and quantum communication [5]. These applications often require maximally entangled states (that is states of the form $|\varphi_m\rangle = \frac{1}{\sqrt{m}}\sum_{i=1}^{m}|i\rangle_A|i\rangle_B$). However bipartite entanglement shared by two parties may well not be maximal. A number of schemes for obtaining maximally entangled states from non-maximal ones have been investigated both in cases where the starting state is pure (the process is then called

entanglement concentration) and where it is mixed (the process is then called purification) [6, 7, 8, 9]. More generally, schemes have also been considered for manipulating general entangled states to other general entangled states by local operations and classical communication (LOCC) [10, 11]. In this paper we will be concerned only with concentrating the entanglement of pure states. These schemes assume that the distant parties sharing the entangled state are only allowed to perform LOCC. In [12], it was shown that N copies of a non-maximal two particle entangled state can be converted to maximally entangled states by LOCC in the asymptotic limit $N \to \infty$. Subsequently, [13, 14, 15], various people showed how to convert a single copy of a general entangled state to a distribution of known maximally entangled states. In such a scheme we consider the process where, under LOCC, we can convert a general pure state $|\psi_m\rangle$ to a distribution of maximally entangled states $|\varphi_m\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} |i\rangle_A |i\rangle_B$ where $|\varphi_m\rangle$ occurs with probability, p_m. The state $|\varphi_m\rangle$ is given weighting $\log_2 m$ since it corresponds to $\log_2 m$ copies of $|\varphi_2\rangle$ [12, 13]. The average maximal entanglement produced is then $E = \sum_m p_m \log_2 m$. In [14, 15], it was shown that for one copy of a general bipartite pure state in Schmidt form, $|\psi\rangle = \sum_{i=0}^{m-1} \sqrt{\lambda_i} |i\rangle_A |i\rangle_B$ with $\lambda_i \geq \lambda_{i+1}$, the maximum average entanglement that can be concentrated in the form of maximally entangled state is given by

$$E^{\max} = \sum_{i=1}^{m} (\lambda_{i-1} - \lambda_i) i \log_2 i \tag{1}$$

(where $\lambda_m = 0$). Hence, for a pure entangled state shared between two parties, we have a full understanding of how to concentrate entanglement.

In this paper we wish to consider more general schemes in which entanglement may be concentrated. In particular, we will consider entanglement swapping. Entanglement swapping was introduced in [16] (and also in [4]). The idea is that Alice shares an entangled state with Bob and Bob shares another entangled state with Charlie. Bob performs a Bell measurement on his two particles and the result is that Alice and Charlie end up sharing an entangled state. Initially, only the case where Alice and Bob and likewise Bob and Charlie shared maximally entangled states was considered (or in the case of [4] at lease one of the states was maximally entangled allowing teleportation). Later, Bose, Vedral and Knight considered the case [17] where both states are of the form $\alpha|00\rangle + \beta|11\rangle$. They found, perhaps rather surprisingly, that the amount of entanglement concentrated corresponds to the optimal amount which could be concentrated were the state $\alpha|00\rangle + \beta|11\rangle$ shared directly between Alice and Charlie (namely $2|\beta|^2$ where $|\beta| \leq |\alpha|$). Very recently, Shi, Jiang and Guo [18] considered the case where Alice and Bob share the state $\alpha|00\rangle + \beta|11\rangle$ and Bob and Charlie share the state $\alpha'|00\rangle + \beta'|11\rangle$. This time

it turns out that the amount of entanglement concentrated between Alice and Charlie corresponds to that which could be concentrated from the less entangled of these two states were it shared directly between Alice and Charlie. In this case the entanglement concentrated corresponds to that of the weakest link. Entanglement swapping chains have also been considered with impure states [19]. The objective there was to exchange pure entanglement over long distance through noisy channels.

In this paper we will generalise this situation in two respects. We will consider the case where the states shared are of $m \times m$ dimensions and we will consider chains consisting of three or more parties with generalised Bell measurements at all the intermediate locations. We will obtain a result for the average amount of entanglement that can concentrated (in the form of maximally entangled states) between the two parties by this method. We find that for a particular rather broad class of cases (but not in all cases) the average amount of entanglement concentrated corresponds to weakest link in the chain. Namely that in these cases, the average entanglement concentrated is equal to that which could be concentrated were that state which has lowest $E^{\max}$ (as given by equation (1)) shared between the two end parties. Therefore, in these cases the protocol is optimal. However, for other cases we do not know that the protocol is optimal - it is possible that measurements other than generalised Bell measurements could yield a higher average.

In section 2 we review the case where three parties share two 2×2 dimensional states as first considered by [18]. In section 3 we generalise this to $m \times m$ dimensional states and in section 4 we generalise further to the case where a number of parties arranged in a chain share entangled states pairwise along the chain. Finally, in section 5 we consider the case where GHZ states are concentrated from two 2×2 states shared between three parties.

2 2×2 case

Suppose Alice and Bob share a general bipartite entangled state $|\psi^{(1)}\rangle$ and Bob and Charlie share another entangled state $|\psi^{(2)}\rangle$. We can always write these states in Schmidt form:

$$|\psi^{(1)}\rangle = \sqrt{\lambda_0^{(1)}}|00\rangle_{12} + \sqrt{\lambda_1^{(1)}}|11\rangle_{12} \tag{2}$$
$$|\psi^{(2)}\rangle = \sqrt{\lambda_0^{(2)}}|00\rangle_{34} + \sqrt{\lambda_1^{(2)}}|11\rangle_{34} \tag{3}$$

where the λ's are real and non-negative and satisfy $\sum_{j=0}^{1} \lambda_j^{(i)} = 1$ for $i = 1, 2$ (normalisation) and are taken to be ordered so that $\lambda_0^{(n)} \geq \lambda_1^{(n)}$ for $n = 1, 2$. Also, for simplicity,

we assume that $\lambda_0^{(1)} \geq \lambda_0^{(2)}$ (this means that the state Alice and Bob share has less than or equal entanglement to the state that Bob and Charlie share). As shown in Figure 1, particles 1 and 4 are spatially separated.

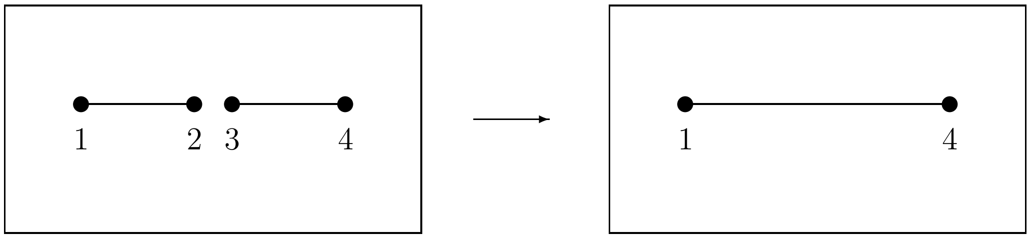


Figure 1: The swapping of entanglement of two bipartite states is shown. A Bell measurement is made on particles 2 and 3.

Bob performs Bell measurement on his particles 2 and 3 onto the basis, $\frac{1}{\sqrt{2}}(|00\rangle_{23} \pm |11\rangle_{23})$ and $\frac{1}{\sqrt{2}}(|10\rangle_{23} \pm |10\rangle_{23})$. Then the particles 1 and 4 for Alice and Charlie are projected onto

$$|\psi\rangle_{14}^{\pm} = P_1^{-1/2} \left(\sqrt{\frac{\lambda_0^{(1)}\lambda_0^{(2)}}{2}} |00\rangle_{14} \pm \sqrt{\frac{\lambda_1^{(1)}\lambda_1^{(2)}}{2}} |11\rangle_{14} \right) \tag{4}$$

$$|\phi\rangle_{14}^{\pm} = P_2^{-1/2} \left(\sqrt{\frac{\lambda_0^{(1)}\lambda_1^{(2)}}{2}} |00\rangle_{14} \pm \sqrt{\frac{\lambda_0^{(2)}\lambda_1^{(1)}}{2}} |11\rangle_{14} \right) \tag{5}$$

where P_1 and P_2 are probabilities for getting $|\psi\rangle_{14}^{\pm}$ and $|\phi\rangle_{14}^{\pm}$ respectively which are

$$\begin{aligned} P_1 &= \frac{\lambda_0^{(1)}\lambda_0^{(2)}}{2} + \frac{\lambda_1^{(1)}\lambda_1^{(2)}}{2} \\ P_2 &= \frac{\lambda_0^{(1)}\lambda_1^{(2)}}{2} + \frac{\lambda_0^{(2)}\lambda_1^{(1)}}{2} \end{aligned} \tag{6}$$

It follows from (1) that for a general bipartite two-level state, the maximum average entanglement concentrated in the form of maximally entangled states is twice the square of the lower coefficient, among two of them. In the 2×2 case this is also equal to the maximum probability of obtaining a $|\varphi_2\rangle$ state. Since we assumed $\lambda_0^{(1)} \geq \lambda_0^{(2)} \geq \lambda_1^{(2)} \geq \lambda_1^{(1)}$, it follows $\lambda_0^{(1)}\lambda_0^{(2)} \geq \lambda_1^{(1)}\lambda_1^{(2)}$ and $\lambda_0^{(1)}\lambda_1^{(2)} \geq \lambda_0^{(2)}\lambda_1^{(2)}$. Therefore the probability of getting

maximally entangled state between 1 and 4 with four states in (4) and (5) is

$$P_1\left(\frac{2\lambda_1^{(1)}\lambda_1^{(2)}}{P_1}\right)+P_2\left(\frac{2\lambda_1^{(1)}\lambda_0^{(2)}}{P_2}\right)=2\lambda_1^{(1)}\lambda_1^{(2)}+2\lambda_1^{(1)}\lambda_0^{(2)}=2\lambda_1^{(1)} \tag{7}$$

This result is optimal because $2\lambda_1^{(1)}$ actually corresponds to the maximum probability of Alice and Bob being able to share a maximally entangled state. We see here that Alice and Charlie are able to share as much maximal entanglement as can be extracted from the weakest link in the chain.

The above procedure can be illustrated with the method of area diagrams introduced in [14]. After cancellation of probability with the normalisation constant and taking the $\pm$ state together, (4) and (5) can be put into area diagrams (i) and (ii) of (a) in Figure 2. Each box has unit width, so the height of the box is equal to its area. $E^{\max}$ can be calculated from column (b) of (i) and (ii) by multiplying the area of that box of width i by $\log_2 i$ then sum over all i's (i.e. $i = 1, 2$). $E^{\max}$ can also be calculated by first adding the boxes of (i) and (ii) therefore get (iii) of (a), then calculate from the boxes in (b) of (iii). Once (i) and (ii) are added, the terms $\lambda_0^{(2)}$ and $\lambda_1^{(2)}$ add up to 1 then it is easy to see that $E^{\max}$ is equal to $E^{\max}$ for $|\psi^{(1)}\rangle$ only, which is $2\lambda_1^{(1)}$. This graphical method is very useful in the general $m \times m$ case.

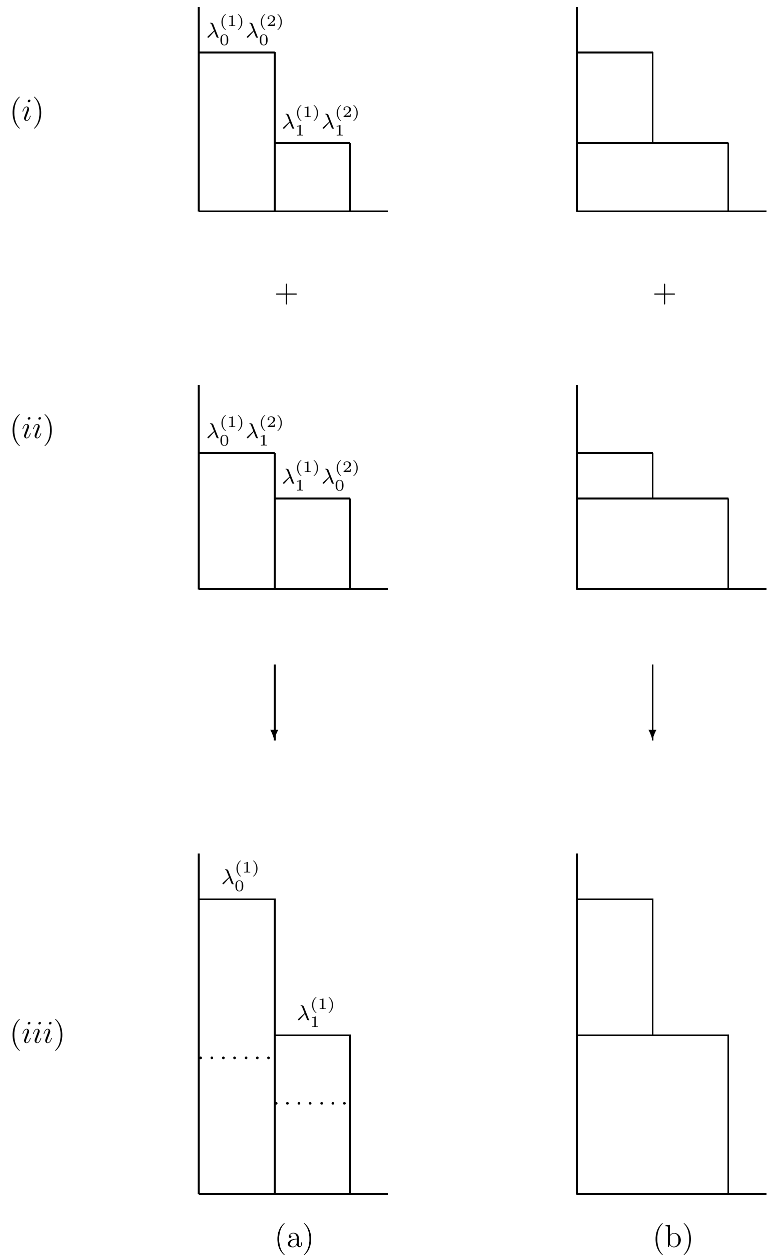


Figure 2: The area diagrams for two 2-level states are shown. The columns have unit width and the number above each column indicates the height.

3 $m \times m$ case

In this section, we generalise the 2×2 case considered in the previous section to the $m \times m$ case, i.e. an entangled state between two m-level systems. Let us consider two entangled pairs of particles (1,2) and (3,4). Assume 1 and 4 are spatially separated as in Figure 1. m-level entangled states for (1,2) and (3,4) are

$$|\psi^{(1)}\rangle_{12} = \sum_{i=0}^{m-1} \sqrt{\lambda_i^{(1)}} |ii\rangle_{12} \tag{8}$$

$$|\psi^{(2)}\rangle_{34} = \sum_{i=0}^{m-1} \sqrt{\lambda_i^{(2)}} |ii\rangle_{34} \tag{9}$$

We can assume that the λ's are ordered such that $\lambda_i^{(n)} \geq \lambda_{i+1}^{(n)}$ for $n = 1, 2$ without loss of generality. Next, particles 2 and 3 are brought together and generalised Bell measurements are performed. We generalise the Bell basis as follows

$$|\psi_\alpha^\beta\rangle_{23} = \frac{1}{\sqrt{m}} \sum_{l=0}^{m-1} e^{((l\beta) \bmod m)2\pi i/m} |l, l+\alpha\rangle_{23} \tag{10}$$

where we abbreviated $|(l+\alpha) \bmod m\rangle$ to $|l+\alpha\rangle$ (which we will assume throughout this paper). For $\alpha, \beta = 0, \cdots, m-1$, it can be checked that $\langle \psi_{\alpha'}^{\beta'} | \psi_\alpha^\beta \rangle = \delta_{\alpha'\alpha} \delta^{\beta'\beta}$, thereby yielding m^2 orthonormal states. With this basis, $|\psi^{(1)}\rangle_{12} \otimes |\psi^{(2)}\rangle_{34}$ is projected onto

$$|\psi_{proj}\rangle = \frac{1}{\sqrt{P_\alpha}} \frac{1}{\sqrt{m}} \sum_{l=0}^{m-1} e^{-((l\beta) \bmod m)2\pi i/m} \sqrt{\lambda_l^{(1)} \lambda_{l+\alpha}^{(2)}} |l, l+\alpha\rangle_{14} \tag{11}$$

with probability P_α, which can be obtained from normalisation. Note that $\bmod m$ arithmetic is assumed for subscripts of λ (which we will assume throughout the remainder of this paper). The (unnormalised) Schmidt coefficients for each $\alpha = 0, \cdots, m-1$ are as follows (β does not make any difference since the phase term disappears),

$$\lambda_0^{(1)} \lambda_{0+\alpha}^{(2)} \quad , \quad \lambda_1^{(1)} \lambda_{1+\alpha}^{(2)} \quad , \cdots , \quad \lambda_{m-1}^{(1)} \lambda_{m-1+\alpha}^{(2)} \tag{12}$$

Now we want to re-order (12) such that the highest value is set equal to Z_0^α, and the next highest is set equal to Z_1^α, and so on until the lowest which is set equal to Z_{m-1}^α. Then the average entanglement obtained from the formula (1) is given as

$$E^{\max} = \sum_{\alpha=0}^{m-1} \sum_{i=1}^{m} (Z_{i-1}^\alpha - Z_i^\alpha) i \log_2 i \tag{13}$$

where $Z_m^\alpha = 0$. Note that the probabilities P_α have cancelled with the normalisation constants in this formula.

Let us consider an example with $m = 3$. From (8) and (9), we have the following states,

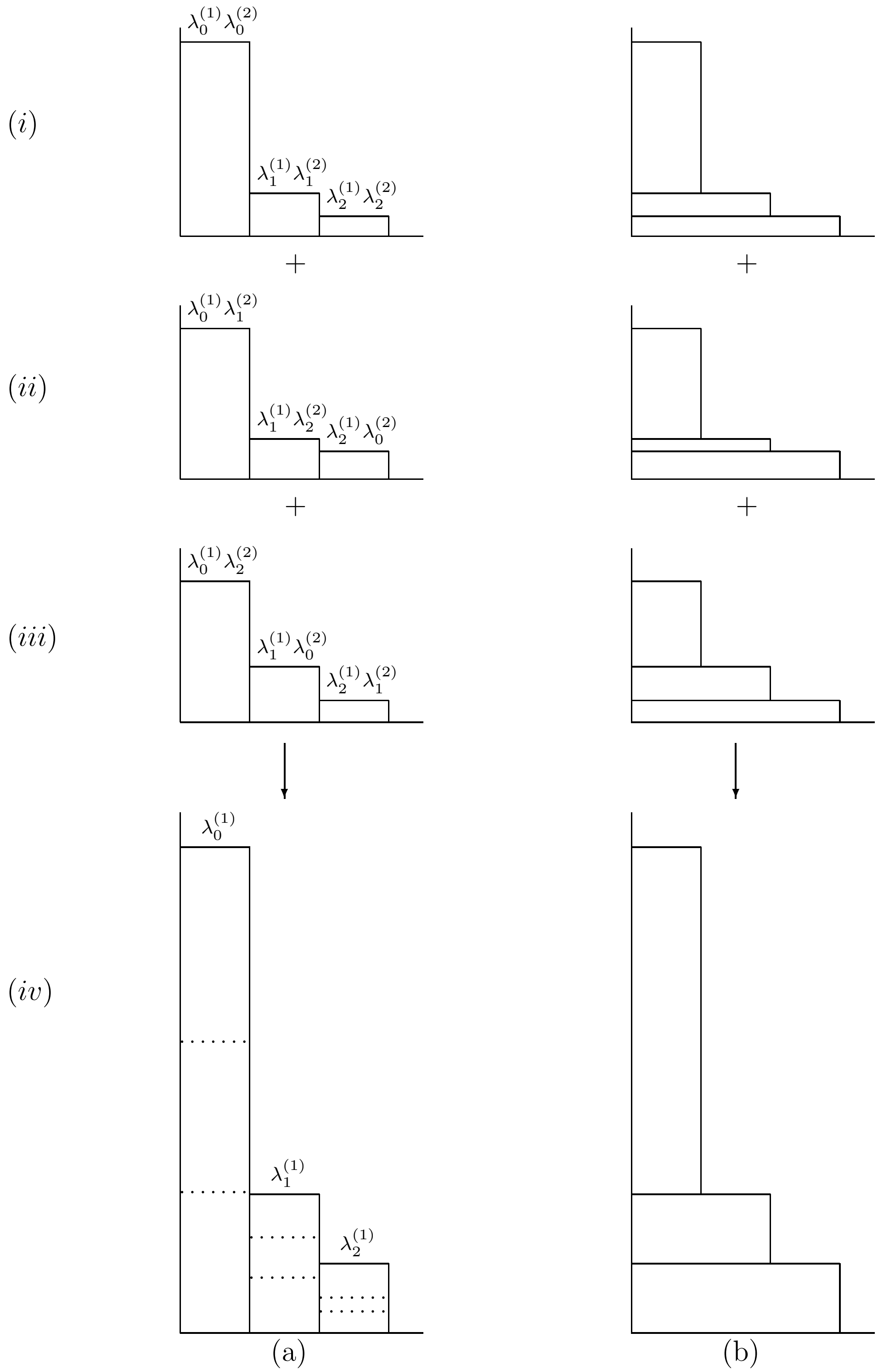
$$|\phi\rangle_{12} = \sqrt{\lambda_0^{(1)}}|00\rangle_{12} + \sqrt{\lambda_1^{(1)}}|11\rangle_{12} + \sqrt{\lambda_2^{(1)}}|22\rangle_{12} \tag{14}$$

$$|\phi\rangle_{34} = \sqrt{\lambda_0^{(2)}}|00\rangle_{34} + \sqrt{\lambda_1^{(2)}}|11\rangle_{34} + \sqrt{\lambda_2^{(2)}}|22\rangle_{34} \tag{15}$$

A generalised Bell measurement is made on particles 2 and 3. The unnormalised Schmidt coefficients given in (12) will be obtained with $m = 3$ and $\alpha = 0, 1, 2$. Suppose we impose the following additional condition on the terms in (12),

$$\begin{aligned} (i)\ \lambda_0^{(1)}\lambda_0^{(2)} &\geq \lambda_1^{(1)}\lambda_1^{(2)} \geq \lambda_2^{(1)}\lambda_2^{(2)} \\ (ii)\ \lambda_0^{(1)}\lambda_1^{(2)} &\geq \lambda_1^{(1)}\lambda_2^{(2)} \geq \lambda_2^{(1)}\lambda_0^{(2)} \\ (iii)\ \lambda_0^{(1)}\lambda_2^{(2)} &\geq \lambda_1^{(1)}\lambda_0^{(2)} \geq \lambda_2^{(1)}\lambda_1^{(2)} \end{aligned} \tag{16}$$

where $(i), (ii)$ and (iii) correspond to $\alpha = 0, 1, 2$ respectively. This means that there is no re-ordering when the Z's are assigned. The Schmidt coefficients in $(i), (ii), (iii)$ can be represented on graphs as shown in (a) of Figure 3. (i) is always true since coefficients of $|\phi\rangle_{12}$ and $|\phi\rangle_{34}$ are ordered. However (ii) and (iii) may not always be the case. Nevertheless, if we assume $(i), (ii)$ and (iii) to hold, then as in the 2 state case, we can add the boxes of $(i), (ii)$ and (iii) and obtain (iv) in Figure 3 and calculate the maximum average entanglement for obtaining a maximal state from (iv) which is just $\sum_{j=1}^{3}(\lambda_{j-1}^{(1)} - \lambda_j^{(1)})j\log_2 j$, the $E^{\max}$ from $|\phi\rangle_{12}$.



(i)
(ii)
(iii)
(iv)
$\lambda_0^{(1)}\lambda_0^{(2)}$
$\lambda_1^{(1)}\lambda_1^{(2)}$
$\lambda_2^{(1)}\lambda_2^{(2)}$
$\lambda_0^{(1)}\lambda_1^{(2)}$
$\lambda_1^{(1)}\lambda_2^{(2)}$
$\lambda_2^{(1)}\lambda_0^{(2)}$
$\lambda_0^{(1)}\lambda_2^{(2)}$
$\lambda_1^{(1)}\lambda_0^{(2)}$
$\lambda_2^{(1)}\lambda_1^{(2)}$
$\lambda_0^{(1)}$
$\lambda_1^{(1)}$
$\lambda_2^{(1)}$
(a)
(b)

Figure 3: The area diagrams for two 3-level states are shown. The columns have unit width and the number above each column indicates the height.

Since the λ's are assumed to be ordered initially, it follows that the columns in (16) are already ordered (e.g. $\lambda_0^{(1)}\lambda_0^{(2)} \geq \lambda_0^{(1)}\lambda_1^{(2)}$) and decrease from top to bottom. Hence, inequalities can be compressed into

$$\lambda_{i_1}^{(1)}\lambda_{i_2}^{(2)} \geq \lambda_{i_1'}^{(1)}\lambda_{i_2'}^{(2)} \tag{17}$$

if and only if

$$i_1 i_2 \leq i_1' i_2' \tag{18}$$

where $i_1 i_2$ is interpreted as a number in base 3 (i.e. it is equal to $3^1 i_1 + 3^0 i_2$). In fact, this can be done for any m where $i_1 i_2$ is now interpreted as a number in base m (i.e. equal to $m^1 i_1 + m^0 i_2$). Then $E^{\max}$ can be obtained from $|\phi\rangle_{12}$ only, i.e. $\sum_{j=1}^{m}(\lambda_{j-1}^{(1)} - \lambda_j^{(1)}) j \log_2 j$. What are the class of λ's which would satisfy the condition in (17) and (18)? One trivial example is when $|\psi^{(2)}\rangle$ is maximally entangled state, so that all $\lambda^{(2)}$'s are $\frac{1}{m}$. In the next section, we generalise the above procedure to the N-chain of m-level entangled states and we give nontrivial class of cases which satisfy the condition in (17) and (18).

In the cases where (17) and (18) are satisfied we have again that the result is optimal and that we can obtain maximal entanglement equal to that extractable from the weakest link in the chain. However, when (17) and (18) do not hold then we do not know that (13) is optimal since it is possible that a different measurement by Bob could yield better results.

4 N-chained case

In this section, we generalise the two $m \times m$ entangled states to N-chained $m \times m$ states. As shown in Fig. 4, there are N such states and the measurements are made on particles $(2,3),(3,4),(5,6),\cdots,(2N-2,2N-1)$, so that we are left with a single entangled pair between the particles 1 and $2N$. We would like to know what the highest average entanglement that can be concentrated in the form of maximally entangled states between 1 and $2N$ is.

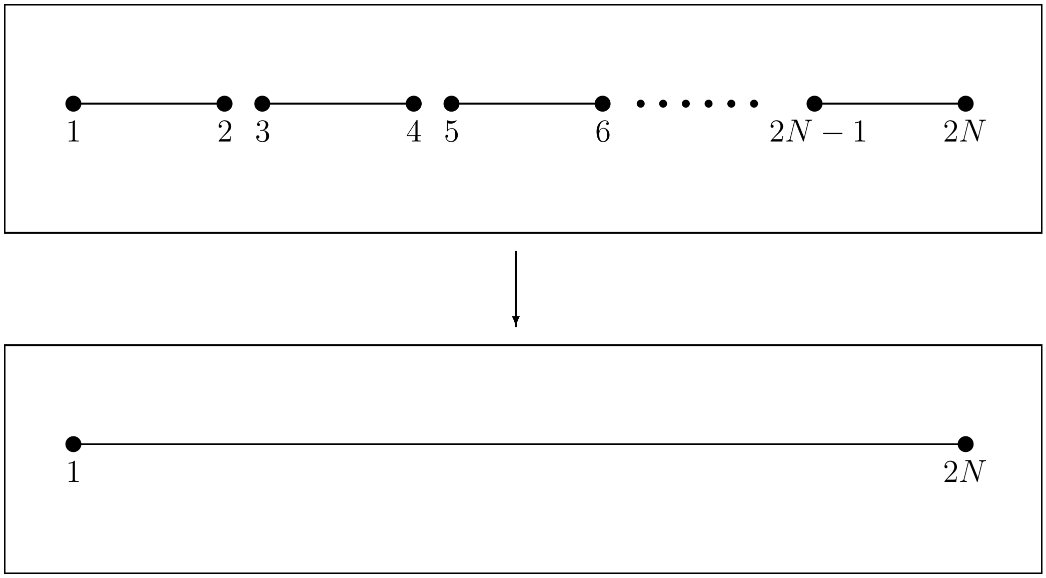


Figure 4: The swapping of entanglement of N-chained states is shown. Generalised Bell measurements are performed at each intermediate site $(2,3),(4,5),\cdots,(2N-2,2N-1)$.

We start with the following $m \times m$ entangled states,

$$
\begin{aligned}
|\varphi^{(1)}\rangle &= \sum_{i=0}^{m-1} \sqrt{\lambda_i^{(1)}}|ii\rangle_{12} \\
|\varphi^{(2)}\rangle &= \sum_{i=0}^{m-1} \sqrt{\lambda_i^{(2)}}|ii\rangle_{34} \\
&\vdots \\
|\varphi^{(N)}\rangle &= \sum_{i=0}^{m-1} \sqrt{\lambda_i^{(N)}}|ii\rangle_{2N-1,2N}
\end{aligned}
\tag{19}
$$

There are $N-1$ intermediate locations which we label by $n = 1$ to $N-1$. At each location, a measurement is made onto the generalised Bell basis $|\psi_{\alpha_n}^{\beta_n}\rangle_{2n,2n+1}$. If we define

$\gamma_n \equiv \sum_{i=1}^{n} \alpha_i$, then $|\psi^{(1)}\rangle \otimes \cdots \otimes |\psi^{(N)}\rangle$ is projected as follows,

$$|\psi_{proj}\rangle = \frac{1}{\sqrt{P_\alpha}} \frac{1}{\sqrt{m}} \sum_{l=0}^{m-1} e^{-((l \sum \beta_n) \mathrm{mod}\, m) 2\pi i/m} \sqrt{\lambda_l^{(1)} \lambda_{l+\gamma_1}^{(2)} \cdots \lambda_{l+\gamma_{N-1}}^{(N)}} |l, l+\gamma_{N-1}\rangle_{1,2N} \quad (20)$$

with probability $P_{\{\alpha\}}$ which can be obtained from normalisation. The unnormalised Schmidt coefficients for each outcome $\{\alpha_i\}$ are,

$$\lambda_0^{(1)} \cdots \lambda_{\gamma_{N-1}}^{(N)} \quad , \quad \lambda_1^{(1)} \cdots \lambda_{1+\gamma_{N-1}}^{(N)} \quad , \quad \cdots \quad , \quad \lambda_{m-1}^{(1)} \cdots \lambda_{m-1+\gamma_{N-1}}^{(N)} \quad (21)$$

(again β does not contribute since the phase terms disappear). As in the previous section, the elements in (21) can be re-ordered (again as $Z_0 \geq \cdots \geq Z_{N-1}$) and $E^{\max}$ can be calculated using the formula in (1) for outcome $\{\alpha_i\}$ then adding all of them up which gives

$$E^{\max} = \sum_{\gamma_1=0}^{m-1} \cdots \sum_{\gamma_{N-1}}^{m-1} \sum_{i=1}^{m} \left(Z_{i-1}^{\gamma_1 \cdots \gamma_{N-1}} - Z_i^{\gamma_1 \cdots \gamma_{N-1}} \right) i \log_2 i \quad (22)$$

between 1 and $2N$. This formula has the curious property that $E^{\max}$ is independent of the particular order that the states (20) are in (since the γ's in (21) take all values).

Let us consider a special case such that,

$$\lambda_{i_1}^{(1)} \lambda_{i_2}^{(2)} \cdots \lambda_{i_N}^{(N)} \geq \lambda_{i'_1}^{(1)} \lambda_{i'_2}^{(2)} \cdots \lambda_{i'_N}^{(N)} \quad (23)$$

if and only if

$$i_1 i_2 \cdots i_N \leq i'_1 i'_2 \cdots i'_N \quad (24)$$

where $i_1 \cdots i_N$ is an integer in base m (i.e. $i_1 m^{N-1} + i_2 m^{N-2} + \cdots + i_N m^0$). Then, by employing similar graphical reasoning to that in the previous section, $E^{\max}$ can be obtained from $|\varphi^{(1)}\rangle$ only, i.e.

$$E^{\max} = \sum_{i=1}^{m} (\lambda_{i-1}^{(1)} - \lambda_i^{(1)}) i \log_2 i \quad (25)$$

In such cases we obtain maximal entanglement extractable from the weakest link. In this case, the weakest link is $|\varphi^{(1)}\rangle$. However since $E^{\max}$ is independent of the order of the states, similar results would apply in the case where one of the other links is the weakest. In general the condition would be

$$\lambda_{i_1}^{(1)} \lambda_{i_2}^{(2)} \cdots \lambda_{i_N}^{(N)} \geq \lambda_{i'_1}^{(1)} \lambda_{i'_2}^{(2)} \cdots \lambda_{i'_N}^{(N)} \quad (26)$$

if and only if

$$\mathrm{Perm}(i_1 i_2 \cdots i_N) \leq \mathrm{Perm}(i'_1 i'_2 \cdots i'_N) \quad (27)$$

where Perm permutes the positions of the entries in the base m number. The weakest link is then that corresponding to the leftmost entry in the string.

In fact, not only is the average entanglement concentrated equal to average entanglement that can be concentrated from the weakest link, but also the distribution of $|\varphi_m\rangle$ states are the same. This follows directly from the area diagrams.

We will now show that there exists a nontrivial class of λ's which satisfy the condition in (23,24). Suppose Λ's are unnormalised λ's, i.e. $\lambda_j^{(l)} = \frac{\Lambda_j^{(l)}}{\left(\sum_{j=0}^{m-1}\Lambda_j^{(l)}\right)}$. With the normalisation factor $L = \left((\sum_{i_1=0}^{m-1}\Lambda_{i_1}^{(1)})\cdots(\sum_{i_N=0}^{m-1}\Lambda_N^{i_N})\right)$, the condition in (23) and (24) becomes

$$\frac{\Lambda_{i_1}^{(1)}\cdots\Lambda_{i_N}^{(N)}}{L} \geq \frac{\Lambda_{i'_1}^{(1)}\cdots\Lambda_{i'_N}^{(N)}}{L} \tag{28}$$

if and only if

$$i_1\cdots i_N \leq i'_1\cdots i'_N \tag{29}$$

Taking logarithm with base $b > 1$, of (28), gives

$$\log_b \Lambda_{i_1}^{(1)} + \cdots + \log_b \Lambda_{i_N}^{(N)} \geq \log_b \Lambda_{i'_1}^{(1)} + \cdots \log_b \Lambda_{i'_N}^{(N)} \tag{30}$$

(since $\log_b$ has positive gradient everywhere when $b > 1$). If we define $\bar{i} \equiv (m-1)-i$ then (29) is equivalent to $\bar{i}_1\bar{i}_2\cdots\bar{i}_N \geq \bar{i}'_1\bar{i}'_2\cdots\bar{i}'_N$. Let us take a set of non-negative constants η_n where $n = 1,\cdots,N$ satisfying $\eta_n \geq (m-1)\eta_{n+1}$. Then (29) implies

$$\bar{i}_1\eta_1 + \cdots + \bar{i}_N\eta_N \geq \bar{i}'_1\eta_1 + \cdots + \bar{i}'_N\eta_N \tag{31}$$

Comparing (30) and (31), we can put $\log_b \Lambda_{i_n}^{(n)} = \bar{i}_n\eta_n$. Then

$$\Lambda_{i_n}^{(n)} = b^{\bar{i}_n\eta_n} \tag{32}$$

The λ's can then be found by normalisation. One example is where $\eta_{n-1} \equiv m\eta_n$ and $\eta_N = 1$, then

$$\Lambda_{i_n}^{(n)} = b^{\bar{i}_n m^{N-n}} \tag{33}$$

In this case the λ's decrease more steeply for smaller n and less steeply for larger n and hence $E^{\max}$ increases with n for the links in the chain. Another interesting example is when $\eta_1 = 1$ and $\eta_n = 0$ for $n = 2$ to N. In this case the first entangled state in the chain will be some non-maximally entangled state while the remaining entangled states will all be maximally entangled (since their Λ's are equal to 1). This is the well known situation

in which the entanglement of the first state is successively teleported along the chain so that it is finally shared by the two end parties.

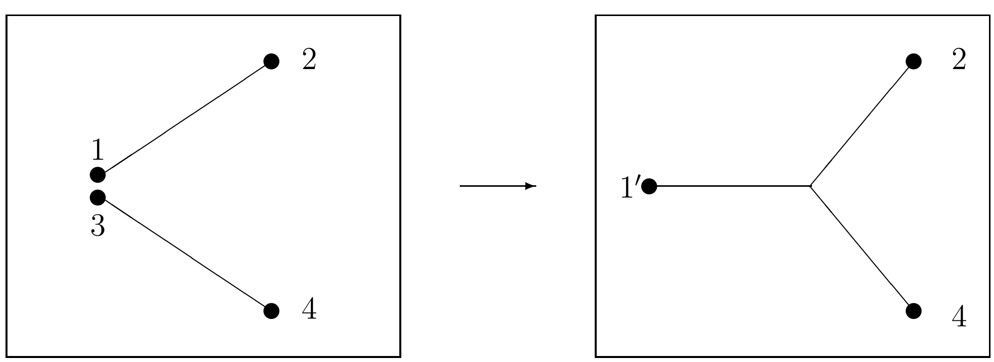
In those cases where (26,27) is satisfied we know that this method is optimal since no concentration protocol could yield better results than that corresponding to the weakest link. However, in those cases where this condition is not satisfied, we do not know that the protocol discussed here is optimal. It is possible that other protocols in which measurements other than generalised Bell measurements are made at the intermediate stages may yield better results.

5 Obtaining GHZ states

So far we have only considered Bell measurements. In this section, we give a simple example that converts two general 2×2 bipartite states into a single GHZ state [20] with some probability. Let Alice and Bob share entangled particles 1 and 2 and Alice and Charlie share 3 and 4 with the following states,

$$\begin{aligned} |\varphi^{(1)}\rangle_{12} &= \sqrt{\lambda_0^{(1)}}|00\rangle_{12} + \sqrt{\lambda_1^{(1)}}|11\rangle_{12} \\ |\varphi^{(2)}\rangle_{34} & \quad \sqrt{\lambda_0^{(2)}}|00\rangle_{34} + \sqrt{\lambda_1^{(2)}}|11\rangle_{34} \end{aligned} \tag{34}$$

where we take $\lambda_0^{(n)} \geq \lambda_0^{(n)}$ for $n = 1, 2$ and we assume that $\lambda_0^{(1)} \geq \lambda_1^{(2)}$.



(a) (b)

Figure 5: The conversion of two bipartite states to a single GHZ state is shown. A degenerate measurement is made on particles 1 and 3.

Alice makes measurements on her particles 1 and 3 of the following (degenerate) operators

$$\begin{aligned} F_1 &= |00\rangle_{13}\langle 00| + |11\rangle_{13}\langle 11| \\ F_2 &= |01\rangle_{13}\langle 01| + |10\rangle_{13}\langle 10| \end{aligned} \tag{35}$$

Then with probability $P_1 = \lambda_0^{(1)}\lambda_0^{(2)} + \lambda_1^{(1)}\lambda_1^{(2)}$ Alice, Bob and Charlie are left with

$$\frac{1}{\sqrt{P_1}}\left(\sqrt{\lambda_0^{(1)}\lambda_0^{(2)}}|0000\rangle_{1324} + \sqrt{\lambda_1^{(1)}\lambda_1^{(2)}}|1111\rangle_{1324}\right) \tag{36}$$

We can redefine the states$|00\rangle_{13}$ and $|11\rangle_{13}$ of particles 1 and 3 which are in Alice's hands to be $|0\rangle_{1'}$ and $|1\rangle_{1'}$ respectively of system $1'$ (system $1'$ consisting of particles 1 and 3). Then the state they share is

$$\frac{1}{\sqrt{P_1}}\left(\sqrt{\lambda_0^{(1)}\lambda_0^{(2)}}|000\rangle_{1'24} + \sqrt{\lambda_1^{(1)}\lambda_1^{(2)}}|111\rangle_{1'24}\right) \tag{37}$$

Also with probability $P_2 = \lambda_0^{(1)}\lambda_1^{(2)} + \lambda_1^{(1)}\lambda_0^{(2)}$, the following state is obtained:

$$\frac{1}{\sqrt{P_2}}\left(\sqrt{\lambda_0^{(1)}\lambda_1^{(2)}}|0101\rangle_{1324} + \sqrt{\lambda_1^{(1)}\lambda_0^{(2)}}|1010\rangle_{1324}\right) \tag{38}$$

Again we can redefine $|01\rangle_{13} \equiv |2\rangle_{1'}$ and $|10\rangle_{13} \equiv |3\rangle_{1'}$. so the state is

$$\frac{1}{\sqrt{P_2}}\left(\sqrt{\lambda_0^{(1)}\lambda_1^{(2)}}|201\rangle_{1324} + \sqrt{\lambda_1^{(1)}\lambda_0^{(2)}}|310\rangle_{1324}\right) \tag{39}$$

It is possible to obtain a GHZ state from either of these two states by performing a local filtering operation. The probability of this being successful is equal to the twice the square of the smaller coefficient in the expansion. Since $\sqrt{\lambda_0^{(1)}\lambda_0^{(2)}} \geq \sqrt{\lambda_1^{(1)}\lambda_1^{(2)}}$ and $\sqrt{\lambda_0^{(1)}\lambda_1^{(2)}} \geq \sqrt{\lambda_1^{(1)}\lambda_0^{(2)}}$, the probability of obtaining a GHZ state is

$$\begin{aligned} &\Rightarrow \quad 2P_1\frac{\lambda_1^{(1)}\lambda_1^{(2)}}{P_1} + 2P_2\frac{\lambda_1^{(1)}\lambda_0^{(2)}}{P_2} \\ &\Rightarrow \quad 2\lambda_1^{(1)} \end{aligned} \tag{40}$$

To see that this is optimal suppose we could do better. That is suppose the parties can obtain a GHZ with higher probability than $2\lambda_1^{(1)}$. Once they have this GHZ Charlie could make a measurement on his particle which collapses the other two particles into 2×2 dimensional maximally entangled state thus obtaining such a state with probability higher than $2\lambda_1^{(1)}$ but this contradicts equation (1) when applied to Alice and Bob.

6 Conclusions

In this paper we have generalised entanglement swapping to a chain of arbitrary pure entangled states in arbitrary dimensions with generalised Bell measurements. For a chain consisting of two 2×2 dimensional entangled states it has been shown that the entanglement concentrated between the two ends is equal to that concentratable from the weakest link in the chain. For longer chains and/or for higher dimensions this result does not hold. However, there does exist a broad class of cases in which the entanglement concentrated is equal to that concentratable from the weakest link. This proves that, in these cases, entanglement swapping (with generalised Bell measurements) is an optimal way of concentrating entanglement. However, we do not know that it is optimal in general. Nevertheless it interesting is that there is a much richer structure when we go to higher dimensions and longer chains.

In this paper we have addressed the matter of entanglement manipulation by LOCC in a particular setting. Namely where we have chains of pure bipartite entangled states and we wish to concentrate to maximally entangled states. The most general setting would be where a number of parties share a general entangled state and want to manipulate it by LOCC to some other state or distribution of states. However, there are many restricted situations (such as a chain of bipartite pure states as considered here) that fall short of this most general setting. By considering other restricted cases we may hope to gain a deeper understanding general entanglement manipulation. The example in section 5 can be considered as work in this direction.

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